

# Gröbner-Shirshov Bases for Lie Algebras: after A. I. Shirshov

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**Abstract:** In this paper, we review Shirshov’s method for free Lie algebras invented by him in 1962 [17] which is now called the Gröbner-Shirshov bases theory.

**Key words:** Lie algebra; Lyndon-Shirshov word; Gröbner-Shirshov basis.

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## 1 Introduction

What is now called the Gröbner-Shirshov method for Lie algebras invented by A. I. Shirshov in 1962 [17]. Actually, that paper based on his paper [15] when Shirshov invented a new linear basis for a free Lie algebra which is now called Lyndon-Shirshov basis (it was defined independently in the paper [9] in the same year). We remark that Lyndon-Shirshov basis is a particular case of a series of bases of a free Lie algebra invented by A. I. Shirshov in his Candidate Science Thesis (Moscow State University, 1953, and his adviser was A. G. Kurosh) and published in 1962 [16] (cf. [13] where these bases are called Hall Bases). We now cite the Zbl review by P. M. Cohn [10] of the paper [15]: “The author varies the usual construction of basis commutators in Lie rings by ordering words lexicographically and not by length. This is used to give a very short proof of the theorem (Magnus [12], Witt [18]) that the Lie algebra obtained from a free associative

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algebra is free. Secondly he derives Friedrich's criterion (this Zbl 52,45) for Lie elements. As the third application he proves that every Lie algebra  $L$  can be embedded in a Lie algebra  $M$  such that in  $M$  any subalgebra of countable dimension is contained in a 2-generated subalgebra." We would like to add that it was also a beginning of Gröbner-Shirshov bases theory for Lie and associative algebras. Lemma 4 of the paper, on special bracketing of a regular (Lyndon-Shirshov) associative word with a fix regular subword, leads to the algorithm of elimination of the leading word of one Lie polynomial in other Lie polynomial, i.e., to the reduction procedure, which is very familiar in the cases of associative and associative-commutative polynomials. Also the above Lemma 4 leads to the crucial notion of composition of two Lie polynomials that will be defined lately in [17].

As for paper [17] itself, it is a fully pioneer paper in the subject. He defines a notion of the composition  $(f, g)_w$  of two Lie (associative) polynomials relative to an associative word  $w$  (it was called lately by  $S$ -polynomial for commutative polynomials by B. Buchberger [7] and [8]). It leads to the algorithm for the construction of a Gröbner-Shirshov basis ( $GSB(S)$ ) of Lie (associative) ideal generated by some set  $S$ : to joint to  $S$  all nontrivial compositions and to eliminate the leading monomials of one polynomial of  $S$  in others. Shirshov proved the lemma, now known as the Composition, or Composition-Diamond Lemma, that if  $f \in Id_{Lie}(S)$ , then  $\bar{f}$ , the leading associative word of  $f$ , has a form  $\bar{f} = u\bar{s}v$ , where  $s \in GSB(S)$ ,  $u, v \in X^*$ . Several years later, Bokut formulated this lemma in the modern form (see [2]). Let  $S$  be a set of Lie polynomials that is complete under composition (i.e., any composition of polynomials of  $S$  is trivial; on the other word,  $S$  is a Gröbner-Shirshov basis). Then if  $f \in Id_{Lie}(S)$ , then  $\bar{f} = u\bar{s}v$ , where  $s \in S$ ,  $u, v \in X^*$ . Of course, by using Shirshov's Composition-Diamond Lemma, it can be easily seen that the set  $Red(S)$  of  $S$ -reduced Lyndon-Shirshov words constitutes a linear basis of the quotient algebra  $Lie(X)/Id(S)$ . The converse is also true.

Explicitly Shirshov's Composition-Diamond Lemma for associative algebra was formulated by L. A. Bokut [3] in 1976 and G. Bergman [1] in 1978.

In this paper, we give a comprehensive proof of Shirshov's Composition-Diamond Lemma for Lie algebras. There is an elementary approach to Gröbner-Shirshov bases theory, including for Lie algebras, in [6]. We use properties of associative Lyndon-Shirshov words (ALSW) and non-associative Lyndon-Shirshov words (NLSW), see for example, [11]. These properties are found by using induction on the length of a word applying Shirshov's elimination procedure of [14] (it is known also as the Lazard or Lazard-Shirshov elimination, cf. [13] and [11]).

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## 2 Preliminaries

We start with the Lyndon-Shirshov associative words.

Let  $X = \{x_i | i \in I\}$  be a well-ordered set with  $x_i > x_p$  if  $i > p$  for any  $i, p \in I$ . Let  $X^*$

be the free monoid generated by  $X$ . For  $u = x_{i_1}x_{i_2} \cdots x_{i_k} \in X^*$ , let

$$\begin{aligned} x_\beta = \min(u) &= \min\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}, \\ \text{fir}(u) &= x_{i_1}, \\ \text{length of } u : |u| &= k. \end{aligned}$$

**Definition 2.1** Let  $u = x_{i_1}x_{i_2} \cdots x_{i_k} \in X^*$ . Then  $u$  is called *Weak-ALSW* if  $\text{fir}(u) > \min(u)$  or  $|u| = 1$ , where *ALSW* means an “associative Lyndon-Shirshov word”.

Let  $u$  be a Weak-ALSW,  $\min(u) = x_\beta$  and  $|u| \geq 2$ . We define

$$X'(u) = \{x_i^j = x_i \underbrace{x_\beta \cdots x_\beta}_j \mid i > \beta, j \geq 0\}.$$

Note that  $x_i^j = x_i \underbrace{x_\beta \cdots x_\beta}_j$  is just a symbol.

Now, we order  $X'(u)$  by the following way:

$$x_{i_1}^{j_1} > x_{i_2}^{j_2} \Leftrightarrow i_1 > i_2 \text{ or } (i_1 = i_2, j_2 > j_1).$$

Suppose that  $u, v$  are Weak-ALSW's and  $\min(v) \geq \min(u) = x_\beta$ . Then we define

$$v'_u = x_{i_1}^{m_1} \cdots x_{i_t}^{m_t} \text{ in } (X'(u))^* \Leftrightarrow v = x_{i_1} \underbrace{x_\beta \cdots x_\beta}_{m_1} \cdots x_{i_t} \underbrace{x_\beta \cdots x_\beta}_{m_t} \text{ in } X^*,$$

where  $x_{i_j} > x_\beta$ ,  $m_j \in \mathbb{N}$ ,  $1 \leq j \leq t$ . For the sake of simpler notation, we use  $u'$  instead of  $v'_u$ .

Throughout Section 2 and 3, we assume that  $x_1 < x_2 < x_3 < \cdots$ .

**Example 2.2** Let  $u = x_2x_1, v = x_3x_2$ . Then  $v'_u = x_3^0x_2^0$ ,  $v' = x_3^1$ .

The following lemma is obvious.

**Lemma 2.3** Let  $u$  be a Weak-ALSW,  $x_\beta = \min(u)$ ,  $u = vw$ ,  $v, w \neq 1$  and  $w \neq x_\beta w_1$ . Then  $u' = v'_u w'_u$ .

**Example 2.4** Let  $v = x_3x_2x_1, w = x_2x_2$  and  $u = vw = x_3x_2x_1x_2x_2$ . Then  $u' = x_3^0x_2^1x_2^0x_2^0$ ,  $v'_u = x_3^0x_2^1$ ,  $w'_u = x_2^0x_2^0$  and  $u' = v'_u w'_u$ .

Recall that without specific explanation, we always use the lexicographic order both on  $(X'(u))^*$  and  $X^*$  (i.e.,  $w > wt$  if  $t \neq 1$  and  $zx_it_1 > zx_jt_2$  if  $x_i > x_j$ ).

**Lemma 2.5** Let  $u, v$  be Weak-ALSW's with  $|v| \geq 2$ . Then  $u > v \Leftrightarrow u'_{uv} > v'_{uv}$ .

*Proof.* Let  $x_\beta = \min(uv)$ . Assume that  $u > v$ . Then there are two cases to consider.

Case 1:

$$\begin{aligned} u &= x_{i_1} \underbrace{x_\beta \cdots x_\beta}_{l_1} \cdots x_{i_{s-1}} \underbrace{x_\beta \cdots x_\beta}_{l_{s-1}} x_{i_s} \underbrace{x_\beta \cdots x_\beta}_{l_s} \cdots \\ v &= x_{i_1} \underbrace{x_\beta \cdots x_\beta}_{l_1} \cdots x_{i_{s-1}} \underbrace{x_\beta \cdots x_\beta}_{l_{s-1}} yz \cdots, \quad \text{where } x_{i_s} > y. \end{aligned}$$

(a) If  $y = x_\beta$ , then

$$\begin{aligned} u'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_{s-2}}^{l_{s-2}} x_{i_{s-1}}^{l_{s-1}} x_{i_s}^{l_s} \cdots \\ v'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_{s-2}}^{l_{s-2}} x_{i_{s-1}}^{l'_{s-1}} \cdots, \quad \text{where } l'_{s-1} > l_{s-1}. \end{aligned}$$

So,  $u'_{uv} > v'_{uv}$ .

(b) If  $y > x_\beta$ , then

$$\begin{aligned} u'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_{s-1}}^{l_{s-1}} x_{i_s}^{l_s} \cdots \\ v'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_{s-1}}^{l_{s-1}} y^n \cdots. \end{aligned}$$

So,  $u'_{uv} > v'_{uv}$ .

Case 2:

$$\begin{aligned} u &= x_{i_1} \underbrace{x_\beta \cdots x_\beta}_{l_1} \cdots x_{i_s} \underbrace{x_\beta \cdots x_\beta}_{l_s} \\ v &= x_{i_1} \underbrace{x_\beta \cdots x_\beta}_{l_1} \cdots x_{i_s} \underbrace{x_\beta \cdots x_\beta}_{l_s} yz \cdots. \end{aligned}$$

(a) If  $y = x_\beta$ , then

$$\begin{aligned} u'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_s}^{l_s} \\ v'_{uv} &= x_{i_1}^{l_1} \cdots x_{i_s}^{l'_s} \cdots, \quad \text{where } l'_s > l_s. \end{aligned}$$

So,  $u'_{uv} > v'_{uv}$ .

(b) If  $y > x_\beta$ , then  $v'_{uv} = u'_{uv} y^n \cdots$  and so,  $u'_{uv} > v'_{uv}$ .

Conversely, assume that  $u'_{uv} > v'_{uv}$ . We will prove that  $u > v$ . There are also two cases to consider.

Case 1:  $u'_{uv} = x_{i_1}^{l_1} \cdots x_{i_s}^{l_s}$ ,  $v'_{uv} = x_{i_1}^{l_1} \cdots x_{i_s}^{l'_s} y^n \cdots$ .

Case 2:  $u'_{uv} = x_{i_1}^{l_1} \cdots x_{i_{s-1}}^{l_{s-1}} x_{i_s}^{l_s} \cdots$ ,  $v'_{uv} = x_{i_1}^{l_1} \cdots x_{i_{s-1}}^{l_{s-1}} x_{i'_s}^{l'_s} \cdots$ , where  $x_{i_s} > x_{i'_s}$  or  $(x_{i_s} = x_{i'_s} \text{ and } l'_s > l_s)$ .

In both cases, it is clear that  $u > v$ .  $\square$

**Definition 2.6** Let  $u \in X^*$ . Then  $u$  is called an ALSW if

$$(\forall v, w \in X^*, v, w \neq 1) \quad u = vw \Rightarrow vw > wv.$$

**Remark:** Let  $u, v \in X^*$  and the  $v'_u \in (X'(u))^*$  be as before. We denote by  $|v|$  the length of  $v$  in  $X^*$  and  $|v'_u|_{X'}$  the length of  $v'_u$  in  $(X'(u))^*$ .

**Lemma 2.7** Let  $u$  be a Weak-ALSW with  $|u| \geq 2$ . Then  $u$  is an ALSW in  $X^*$  if and only if  $u'$  is an ALSW in  $(X'(u))^*$ .

*Proof.* “ $\Rightarrow$ ” If  $|u'|_{X'} = 1$ , then  $u'$  is an ALSW. Suppose that  $|u'|_{X'} > 1$  and  $u' = v'_u w'_u$ . Then  $u = vw$ . Since  $u$  is an ALSW,  $vw > wv$  which implies  $(vw)'_u > (wv)'_u$  by Lemma 2.5. Therefore, by Lemma 2.3,  $v'_u w'_u > w'_u v'_u$  and so,  $u'$  is an ALSW.

“ $\Leftarrow$ ” Let  $u = vw$  and  $x_\beta = \min(u)$ . If  $\text{fir}(w) = x_\beta$ , then  $vw > wv$ . If  $\text{fir}(w) \neq x_\beta$ , then

$$u' = v'_u w'_u \Rightarrow v'_u w'_u > w'_u v'_u \Rightarrow (vw)'_u > (wv)'_u \Rightarrow vw > wv.$$

Hence,  $u$  is an ALSW.

**Remark:** For a Weak-ALSW  $u$ , it is clear that  $|u'|_{X'} < |u|$  if  $|u| > 1$ . For an ALSW  $u$ , we denote by  $u'' = (u')'$  and  $u^{(k)} = (u')^{(k-1)}$  for  $k > 0$  generally. From this, it follows that  $X^k(u) = X^{k-1}(u')$ .

**Lemma 2.8** For  $u \in X^*$ ,  $u$  is an ALSW if and only if  $(\exists k \geq 0)$ , s.t.,  $|u^{(k)}|_{X^k(u)} = 1$ .

*Proof.* We apply induction on  $|u|$ . If  $|u| = 1$ , then there is nothing to do. Assume that  $|u| > 1$ . Since  $|u'|_{X'} < |u|$  and

$$|u^{(k)}|_{X^k(u)} = |(u')^{(k-1)}|_{X^{k-1}(u')},$$

by induction and by Lemma 2.7, the result follows.  $\square$

**Example 2.9** Let  $u = x_5 x_4 x_5 x_3$ . Then

$$u' = x_5^0 x_4^0 x_5^1, \quad u'' = (x_5^0)^1 (x_5^1)^0 \text{ and } u''' = ((x_5^0)^1)^1.$$

Therefore, by Lemma 2.8,  $u$  is an ALSW.

**Lemma 2.10** Let  $u \in X^*$ . Then  $u$  is an ALSW if and only if

$$(\forall v, w \in X^*, v, w \neq 1) \quad u = vw \Rightarrow u > w.$$

*Proof.* “ $\Rightarrow$ ” Induction on  $|u|$ . If  $|u| = 1$ , then the result clearly holds. Suppose that  $|u| \geq 2$ ,  $x_\beta = \min(u)$  and  $u = vw$ ,  $v, w \neq 1$ . If  $w = x_\beta w_1$ , then  $u > w$ . If  $w \neq x_\beta w_1$ , then  $u' = v'_u w'_u$ . Since  $u'$  is an ALSW, by induction,  $u' > w'_u$ . Hence, by Lemma 2.5,  $u > w$ .

“ $\Leftarrow$ ” Induction on  $|u|$ . If  $|u| = 1$ , then  $u = x_i$  is an ALSW. If  $|u| > 1$  and  $|u'|_{X'} = 1$ , then by Lemma 2.8,  $u$  is an ALSW. If  $|u'|_{X'} > 1$  and  $u' = v'_u w'_u$ , then  $u' > w'_u$  follows from  $u > w$ . By induction,  $u'$  is an ALSW. Hence, by Lemma 2.7,  $u$  is an ALSW.  $\square$

**Lemma 2.11** Suppose that  $u$  is an ALSW,  $x_\beta = \min(u)$  and  $|u| > 1$ . Then  $ux_\beta$  is an ALSW.

*Proof.* Follows from Lemma 2.10.  $\square$

**Lemma 2.12** Let  $u$  and  $v$  be ALSW's. Then  $uv$  is an ALSW if and only if  $u > v$ .

*Proof.* “ $\implies$ ” Suppose that  $uv$  is an ALSW. Then, by Lemma 2.10,  $u > uv > v$ .

“ $\impliedby$ ” We use induction on  $|uv|$ . Suppose that  $u > v$ . If  $|uv| = 2$  or  $v = x_\beta = \min(uv)$ , then the result is obvious. Otherwise, we can get that  $u'_{uv} > v'_{uv}$ , where  $u'_{uv}, v'_{uv}$  are ALSW's. By induction,  $u'_{uv}v'_{uv} = (uv)'$  is an ALSW and so is  $uv$ .  $\square$

**Lemma 2.13** For any  $u \in X^*$ , there exists a unique decomposition  $u = u_1u_2 \cdots u_k$ , where  $u_i$  is an ALSW,  $1 \leq i \leq k$ , and  $u_1 \leq u_2 \leq \cdots \leq u_k$ .

*Proof.* To prove the existence, we use induction on  $|u|$ . If  $|u| = 1$  then it is trivial. Let  $|u| > 1$  and  $x_\beta = \min(u)$ . If  $u = x_\beta v$ , then  $v$  has the required decomposition and so does  $u$ . Otherwise,  $u$  is a Weak-ALSW. Thus,  $u'$  has the decomposition and so does  $u$ , by Lemma 2.5 and Lemma 2.7.

To prove the uniqueness, we let  $u = u_1 \cdots u_k = w_1 \cdots w_s$  be the decompositions such that  $u_i, w_j$  are ALSW's for any  $i, j$ ;  $u_1 \leq \cdots \leq u_k$  and  $w_1 \leq \cdots \leq w_s$ . If  $u = x_\beta v$ , then  $u_1 = w_1 = x_\beta$  and the result follows from the induction on  $|u|$ . Otherwise,  $u$  is a Weak-ALSW and  $u' = u'_{1u} \cdots u'_{ku} = w'_{1u} \cdots w'_{su}$  are the decompositions of  $u'$ . Now, by induction again, the result follows.  $\square$

**Remark:** In Lemma 2.13, the word  $u_k$  is the longest ALSW end of  $u$ .

**Example 2.14** Let  $u = x_1x_1x_2x_1x_2x_1x_1$ . Then

$$u = \underbrace{x_1}_{u_1} \underbrace{x_1}_{u_2} \underbrace{x_2x_1x_2x_1x_1}_{u_3} = u_1u_2u_3$$

is the decomposition of  $u$ .

**Lemma 2.15** Let  $u$  be an ALSW and  $|u| \geq 2$ . If  $u = vw$ , where  $w$  is the longest ALSW proper end of  $u$ , then  $v$  is an ALSW.

*Proof.* Suppose that  $v$  is not an ALSW. Then, by Lemma 2.13, we can assume that

$$v = v_1v_2 \cdots v_m \quad (m > 1),$$

where each  $v_i$  is an ALSW and  $v_1 \leq v_2 \leq \cdots \leq v_m$ . If  $v_m > w$ , then  $v_mw$  is an ALSW and  $|v_mw| > |w|$ , a contradiction. If  $v_m \leq w$ , then we get another decomposition of  $u$  which contradicts the uniqueness in Lemma 2.13. Thus,  $v$  must be an ALSW.  $\square$

**Example 2.16** Let  $u = x_5x_4x_5x_4x_3x_5x_3$ . Then

$$u = \underbrace{x_5x_4}_v \underbrace{x_5x_4x_5x_3}_w = vw$$

and  $u, v, w$  are all ALSW's.

Now, for an ALSW  $u$ , we introduce two bracketing ways.

One is up-to-down bracketing which is defined inductively by

$$[x_i] = x_i, \quad [u] = [[v][w]],$$

where  $u = vw$  and  $w$  is the longest ALSW proper end of  $u$ .

**Example 2.17** Let  $u = x_2x_2x_1x_1x_2x_1$ . Then

$$u \rightarrow [[x_2x_2x_1x_1][x_2x_1]] \rightarrow [[x_2[x_2x_1x_1]][x_2x_1]] \rightarrow [[x_2[[x_2x_1]x_1]][x_2x_1]].$$

The other is down-to-up bracketing. Let us explain it on a sample word

$$u = x_2x_2x_1x_1x_2x_1.$$

Join the minimal letter  $x_1$  to the previous letters:

$$u \mapsto x_2[x_2x_1]x_1[x_2x_1].$$

Form a new alphabet of the nonassociative words  $x_2$ ,  $[x_2x_1]$  and  $x_1$  ordered lexicographically, i.e.,

$$x_2 > [x_2x_1] > x_1.$$

Join the minimal letter  $x_1$  to the previous letters:

$$x_2[x_2x_1]x_1[x_2x_1] \mapsto x_2[[x_2x_1]x_1][x_2x_1].$$

Form a new alphabet

$$x_2 > [x_2x_1] > [[x_2x_1]x_1].$$

Join the minimal letter  $[[x_2x_1]x_1]$  to the previous letter:

$$x_2[[x_2x_1]x_1][x_2x_1] \mapsto [x_2[[x_2x_1]x_1]][x_2x_1].$$

Form a new alphabet

$$[x_2[[x_2x_1]x_1]] > [x_2x_1].$$

Finally, join the minimal letter  $[x_2x_1]$  to the previous letter:

$$[x_2[[x_2x_1]x_1]][x_2x_1] \mapsto [[x_2[[x_2x_1]x_1]][x_2x_1]] = [u].$$

**Remark:** We denote by  $[ ]$  the down-to-up bracketing and by  $[[ ]]$  the up-to-down bracketing.

**Lemma 2.18**  $[u] = [[u]]$  for any ALSW  $u$ .

*Proof.* We use induction on  $|u|$ . If  $|u| = 1$ , then  $u = x_i$  and  $[x_i] = [[x_i]] = x_i$ . Assume that  $u = vw$ , where  $w$  is the longest ALSW proper end of  $u$ . Then, by Lemma 2.15,  $v$  is an ALSW. If  $w = x_\beta w_1$ , then

$$w = x_\beta \text{ and } v = x_i x_\beta \cdots x_\beta.$$

By induction, we can get  $[v] = [[v]]$ . Hence, in this case,

$$[u] = [[u]] = ([v]x_\beta).$$

If  $w \neq x_\beta w_1$ , then  $u' = v'_u w'_u$ . Suppose that  $v'_u = v'_{1_u} v'_{2_u}$  such that  $v'_{2_u} w'_u$  is an ALSW. Then  $v_2 w$  is an ALSW and  $|v_2 w| > |w|$ , a contradiction. This proves that  $w'_u$  is the longest ALSW proper end of  $u'$ . By induction,  $[v'_u] = [[v'_u]]$  and  $[w'_u] = [[w'_u]]$ . Moreover, by the definition of  $[ ]$  and  $[[ ]]$ , we have

$$[ ] : u \mapsto v'_u w'_u \mapsto [v'_u][w'_u] \mapsto \dots$$

$$[[ ]] : u \rightarrow v'_u w'_u \rightarrow [[v'_u]][[w'_u]] \rightarrow \dots.$$

Therefore,  $[u] = [[u]]$ .  $\square$

### 3 Free Lie algebras

Now we give the definition of a non-associative Lyndon-Shirshov word.

**Definition 3.1** *Let  $<$  be the order on  $X^*$  as before and  $(u)$  a non-associative word. Then  $(u)$  is called a non-associative Lyndon-Shirshov word, denoted by NLSW, if*

- (i)  $u$  is an ALSW,
- (ii) if  $(u) = ((v)(w))$ , then both  $(v)$  and  $(w)$  are NLSW's,
- (iii) in (ii) if  $(v) = ((v_1)(v_2))$ , then  $v_2 \leq w$  in  $X^*$ .

**Remark:** In Definition 3.1 (ii),  $v > w$  by Lemma 2.12.

**Theorem 3.2** *Let  $u$  be an ALSW. Then there exists a unique bracketing way such that  $(u)$  is a NLSW.*

*Proof.* (Existence). Let  $u$  be an ALSW. We will prove that up-to-down bracketing is one of bracketing way such that  $[[u]]$  is a NLSW. Induction on  $|u|$ . If  $|u| = 1$ , then nothing to do. Suppose that  $|u| > 1$  and  $u = vw$  where  $w$  is the longest ALSW proper end of  $u$ . Then,  $[[u]] = [[[v]][[w]]]$ . By induction, both  $[[v]]$  and  $[[w]]$  are NLSW's. Now, we assume that

$$[[v]] = [[v_1]][[v_2]] \text{ and } v_2 > w.$$

Then,  $v_2 w$  is an ALSW, a contradiction. So,  $v_2 \leq w$  and hence,  $[[u]]$  is a NLSW.

(Uniqueness). We assume that  $u$  is an ALSW and  $( )$  is a bracketing way such that  $(u)$  is a NLSW. Then, we have to show  $(u) = [[u]]$ . We use induction on  $|u|$ . If  $|u| = 1$ , then  $(u) = [[u]]$  clearly. Suppose that

$$|u| > 1 \text{ and } u = x_{i_1} x_\beta \cdots x_\beta \cdots x_{i_s} x_\beta \cdots x_\beta,$$

where  $x_{i_j} > x_\beta = \min(u)$ .



Note that if  $v = x_i x_\beta \cdots x_\beta$ ,  $x_i > x_\beta$ , then

$$[[v]] = [[\cdots [[x_i x_\beta] \cdots x_\beta]] x_\beta]$$

is the unique bracketing way such that  $[[v]]$  is a NLSW. According to the definition of NLSW, any associative word in a bracket must be ALSW. Hence,

$$\begin{aligned} (u) &= ((x_{i_1} x_\beta \cdots x_\beta)(x_{i_2} x_\beta \cdots x_\beta) \cdots (x_{i_s} x_\beta \cdots x_\beta)) \\ &= [[[[x_{i_1} x_\beta \cdots x_\beta][[x_{i_2} x_\beta \cdots x_\beta]] \cdots [x_{i_s} x_\beta \cdots x_\beta]]]]. \end{aligned}$$

By induction,  $(u') = [[u']]$  and therefore,  $[[u]] = [[u']] = (u') = (u)$ .  $\square$

Let  $X^{**}$  be the set of all non-associative words  $(u)$  in  $X$ . If  $(u)$  is a NLSW, then we denote it by  $[u]$ .

From now on, let  $k\langle X \rangle$  be the free associative algebra generated by  $X$ . We consider  $( )$  as Lie bracket in  $k\langle X \rangle$ , i.e., for any  $a, b \in k\langle X \rangle$ ,  $(ab) = ab - ba$ . Denote by  $Lie(X)$  the subLie-algebra of  $k\langle X \rangle$  generated by  $X$ .

Given a polynomial  $f \in k\langle X \rangle$ , it has the leading word  $\bar{f} \in X^*$  according to the above order on  $X^*$  such that

$$f = \sum_{u \in X^*} f(u)u = \alpha \bar{f} + \sum \alpha_i u_i,$$

where  $\bar{f}$ ,  $u_i \in X^*$ ,  $\bar{f} > u_i$ ,  $\alpha, \alpha_i, f(u) \in k$ . We call  $\bar{f}$  the leading term of  $f$ . Denote the set  $\{u | f(u) \neq 0\}$  by  $supp f$  and  $deg(f)$  by  $|\bar{f}|$ .  $f$  is called monic if  $\alpha = 1$ .

Note that if  $|u| = |v|$  and  $u < v$ , then the lexicographic order which we use before is the same as the degree-lexicographic order on  $X^*$ .

**Theorem 3.3** *Let the order  $<$  be as before. Then, for any  $(u) \in X^{**}$ ,  $(u)$  has a representation:*

$$(u) = \sum \alpha_i [u_i],$$

where each  $\alpha_i \in k$ ,  $[u_i]$  is a NLSW and  $|u_i| = |u|$ . Even more, if  $(u) = ([v][w])$ , then  $u_i > \min\{v, w\}$ .

*Proof.* Induction on  $|u|$ . If  $|u| = 1$ , then  $(u) = [u]$  and the result holds. Suppose that  $|u| > 1$  and  $(u) = ((v)(w))$ . Then, by induction,

$$(v) = \sum \alpha_i [v_i] \quad \text{and} \quad (w) = \sum \beta_j [w_j],$$

where  $\alpha_i, \beta_j \in k$ ,  $[v_i], [w_j]$  are NLSW's,  $|v_i| = |v|$  and  $|w_j| = |w|$ . Without loss of generality, we may assume that  $(u) = ([v][w])$  with  $v > w$  because of  $([v][w]) = -([w][v])$ . If  $|v| = 1$ , then

$$(u) = ([v][w])$$

is a NLSW. Suppose that  $|v| > 1$  and  $[v] = [[v_1][v_2]]$ .

There are two subcases

(a) If  $v_2 \leq w$ , then  $(u) = ((([v_1][v_2])[w]))$  is a NLSW.

(b) If  $v_2 > w$ , then

$$(u) = (([v_1][v_2])[w]) = (([v_1][w])[v_2]) + ([v_1]([v_2][w])).$$

By induction,

$$\begin{aligned}([v_1][w]) &= \sum \gamma_i[t_i], \quad t_i > \min\{v_1, w\} = w \\([v_2][w]) &= \sum \gamma_j'[t'_j], \quad t'_j > \min\{v_2, w\} = w.\end{aligned}$$

Then,

$$(u) = \sum \gamma_i([t_i][v_2]) + \sum \gamma_j'([v_1][t'_j]).$$

By noting that

$$\min\{t_i, v_2\} \quad \text{and} \quad \min\{t'_j, v_1\} > \min\{v, w\} = w,$$

the result follows from the inverse induction on  $\min\{v, w\}$ .  $\square$

**Example 3.4** Let  $(u) = (((x_3x_2)(x_2x_1))(x_2x_1x_1))$ . Then

$$\begin{aligned}(u) &= (((x_3(x_2x_1))x_2)(x_2x_1x_1)) + ((x_3(x_2(x_2x_1)))(x_2x_1x_1)), \\(((x_3(x_2x_1))x_2)(x_2x_1x_1)) &= (((x_3(x_2x_1))(x_2x_1x_1))x_2) + ((x_3(x_2x_1))(x_2(x_2x_1x_1))) \\&= (((x_3(x_2x_1x_1))(x_2x_1))x_2) + ((x_3((x_2x_1)(x_2x_1x_1)))(x_2) \\&\quad + ((x_3(x_2x_1))(x_2(x_2x_1x_1)))), \\((x_3(x_2(x_2x_1)))(x_2x_1x_1)) &= ((x_3(x_2x_1x_1))(x_2(x_2x_1))) + (x_3((x_2(x_2x_1))(x_2x_1x_1))) \\&= ((x_3(x_2x_1x_1))(x_2(x_2x_1))) + (x_3((x_2(x_2x_1x_1))(x_2x_1)) \\&\quad + (x_3(x_2((x_2x_1)(x_2x_1x_1)))),\end{aligned}$$

and hence,

$$\begin{aligned}(u) &= (((x_3(x_2x_1x_1))(x_2x_1))x_2) + ((x_3((x_2x_1)(x_2x_1x_1)))(x_2) \\&\quad + ((x_3(x_2x_1))(x_2(x_2x_1x_1))) + ((x_3(x_2x_1x_1))(x_2(x_2x_1))) \\&\quad + (x_3((x_2(x_2x_1x_1))(x_2x_1)) + (x_3(x_2((x_2x_1)(x_2x_1x_1))))\end{aligned}$$

is a linear combination of NLSW's.

**Lemma 3.5** Let  $[u]$  be a NLSW. Then  $\overline{[u]} = u$ .

*Proof.* We use induction on  $|u|$ . If  $|u| = 1$ , then the result holds immediately. Let  $|u| > 1$  and  $[u] = [[v][w]]$ . Then, by induction,  $\overline{[v]} = v$  and  $\overline{[w]} = w$ . Suppose that

$$[v] = v + \sum_{v_i < v} \alpha_i v_i, \quad [w] = w + \sum_{w_j < w} \beta_j w_j,$$

where  $\alpha_i, \beta_j \in k$ ,  $v, v_i, w, w_j \in X^*$ . It is easy to see that  $|v_i| = |v|$  and  $|w_j| = |w|$  for any  $i, j$ . Then,

$$\begin{aligned}
[u] &= [(v + \sum_{v_i < v} \alpha_i v_i)(w + \sum_{w_j < w} \beta_j w_j)] \\
&= (v + \sum_{v_i < v} \alpha_i v_i)(w + \sum_{w_j < w} \beta_j w_j) - (w + \sum_{w_j < w} \beta_j w_j)(v + \sum_{v_i < v} \alpha_i v_i) \\
&= vw + \sum_{w_j < w} \beta_j v w_j + \sum_{v_i < v} \alpha_i v_i w + \sum \alpha_i \beta_j v_i w_j \\
&\quad - wv - \sum_{w_j < w} \beta_j w_j v - \sum_{v_i < v} \alpha_i w v_i - \sum \beta_j \alpha_i w_j v_i.
\end{aligned}$$

Since

$$vw > vw_j, v_i w, v_i w_j, wv \text{ and } wv > wv_i, w_j v, w_j v_i,$$

we have,  $\overline{[u]} = u$ .  $\square$

**Remark.** By the proof of Lemma 3.5, if we consider  $[u]$  as a polynomial in  $k\langle X \rangle$ , then each  $r \in \text{supp}([u])$  has the same length as  $u$ , moreover,  $\text{cont}(r) = \text{cont}(u)$ , where, for example,  $\text{cont}(u) = \{x_{i_1}, \dots, x_{i_t}\}$  if  $u = x_{i_1} \cdots x_{i_t} \in X^*$ .

**Lemma 3.6** *NLSW's are  $k$ -independent.*

*Proof.* Suppose

$$\sum_{i=1}^k \alpha_i [u_i] = 0,$$

where each  $\alpha_i \in k$ ,  $[u_i]$  is a NLSW and  $u_1 > u_2 > \cdots > u_k$ . If  $\alpha_1 \neq 0$ , then  $\sum_i \alpha_i \overline{[u_i]} = u_1 \neq 0$ , a contradiction. Then, all  $\alpha_i$  must be 0.  $\square$

By Theorem 3.3 and Lemma 3.6, we have the following corollary.

**Corollary 3.7** *NLSW's are linear basis of  $\text{Lie}(X)$ .*

From Corollary 3.7 and Lemma 3.5, we have

**Corollary 3.8** *For any  $f \in \text{Lie}(X)$ ,  $\bar{f}$  is an ALSW.*

**Theorem 3.9**  *$\text{Lie}(X)$  is the free Lie algebra generated by  $X$ .*

*Proof.* Let  $L$  be a Lie algebra and  $f : X \longrightarrow L$  a mapping. Then, we define a mapping

$$\bar{f} : \text{Lie}(X) \longrightarrow L; [x_{i_1} \cdots x_{i_n}] \longmapsto [f(x_{i_1}) \cdots f(x_{i_n})],$$

where  $[x_{i_1} \cdots x_{i_n}]$  is NLSW. It is easy to check  $\bar{f}$  is a unique Lie homomorphism such that  $\bar{f}i = f$ .

$$\begin{array}{ccc}
X & \xrightarrow{i} & Lie(X) \\
f \downarrow & \nearrow \exists! \bar{f} & \\
L & & 
\end{array}$$

□

The following theorem plays a key role in proving the Composition-Diamond lemma for Lie algebras (see Theorem 5.8).

**Theorem 3.10** (A. I. Shirshov [15]) *Let  $u, v$  be ALSW's,  $u = avb$ ,  $a, b \in X^*$ . Then*

(i)  $[u] = [a[v]c]d$ , where  $b = cd$ ,  $c, d \in X^*$ .

(ii) Let

$$[u]_v = [u]|_{[vc] \mapsto [[[v][c_1]] \cdots [c_k]]}, \quad (1)$$

where  $c = c_1 \cdots c_k$ ,  $c_j$  is an ALSW and  $c_1 \leq c_2 \leq \cdots \leq c_k$ . Then, in  $k\langle X \rangle$ ,

$$\overline{[u]_v} = u.$$

Moreover,

$$[u]_v = a[v]b + \sum_i \alpha_i a_i [v]b_i,$$

where each  $\alpha_i \in k$  and  $a_i v b_i < avb$ .

*Proof.* (i) Induction on  $|u|$ . If  $|u| = 1$ , then  $u = v = x_i$  and the result holds. Assume that  $|u| > 1$ . If  $v = x_i$ , then  $[u] = [a[x_i]d]$  and the result holds. Now, we consider the case of  $|v| > 1$ . Let  $x_\beta = \min(u)$  and  $b = x_\beta^e \tilde{b}$ , where  $e \geq 0$  and  $\text{fir}(\tilde{b}) \neq x_\beta$ . Then

$$u = avb = avx_\beta^e \tilde{b} = a\tilde{v}\tilde{b},$$

where  $\tilde{v} = vx_\beta^e$  is also an ALSW, by Lemma 2.11. Then, by induction, for  $u' = a'_u \tilde{v}'_u \tilde{b}'_u$ , we have  $[u'] = [a'_u [\tilde{v}'_u \tilde{c}'_u] d'_u]$ ,  $\tilde{b}'_u = \tilde{c}'_u d'_u$ . By substitution  $x_i^j \mapsto [[x_i x_\beta] \cdots x_\beta]$ , we obtain

$$[u] = [a[\tilde{v}\tilde{c}]d] = [a[vx_\beta^e \tilde{c}]d] = [a[v]c]d, \quad \text{where } c = x_\beta^e \tilde{c}.$$

(ii) If  $c = 1$ , then  $[u]_v = [u]$  and the results hold clearly. Otherwise, by Lemma 2.13, we may assume that

$$c = x_\beta \cdots x_\beta c_{l+1} \cdots c_k,$$

where each  $c_i$  is an ALSW and  $x_\beta < c_{l+1} \leq \cdots \leq c_k$ .

Then

$$[u]_v = [u]|_{[vx_\beta^e \tilde{c}] \mapsto [[[v]x_\beta] \cdots x_\beta [c_{l+1}] \cdots [c_k]]} \quad \text{and} \quad [u]_{\tilde{v}} = [u]|_{[\tilde{v}\tilde{c}] \mapsto [[[v][c_{l+1}]] \cdots [c_k]]}.$$

Now, we use induction on  $|u|$ . If  $|u| = 1$ , then this is a trivial case. Suppose that  $|u| > 1$  and  $|v| > 1$ . Then, by (i),

$$u = a\tilde{v}\tilde{c}d, \quad u' = a'_u \tilde{v}'_u \tilde{c}'_u d'_u$$

and by induction,

$$[u']_{\tilde{v}'_u} = a'_u[\tilde{v}'_u]\tilde{c}'_u d'_u + \sum_{i \in I_1} \alpha_i a'_{i_u}[\tilde{v}'_u] b'_{i_u},$$

where each  $a'_{i_u} \tilde{v}'_u b'_{i_u} < u'$ . Now, it is easy to check that

$$[[x_i x_\beta] \cdots x_\beta] = \sum_{m \geq 0} (-1)^m \binom{j}{m} x_\beta^m x_i x_\beta^{j-m} \quad \text{and} \quad x_\beta^m x_i x_\beta^{j-m} < x_i x_\beta^j \quad (m > 0).$$

Now, by substitution  $x_i^j \mapsto [[x_i x_\beta] \cdots x_\beta]$ , we obtain

$$[u]_{\tilde{v}} = a[\tilde{v}] \tilde{c} d + \sum_{i \in I_2} \alpha_i a_i[\tilde{v}] b_i,$$

where each  $a_i \tilde{v} b_i < a \tilde{v} \tilde{c} d$ . Also, by substitution  $[\tilde{v}] \mapsto [[[v] x_\beta] \cdots x_\beta]$ , we have

$$[u]_v = a[v] x_\beta^e \tilde{c} d + \sum_{j \in I} \beta_j a_j[v] b_j = a[v] b + \sum_{j \in I} \beta_j a_j[v] b_j,$$

where each  $a_j v b_j < a v b$ .  $\square$

**Remark.** By the proof of Theorem 3.10, if we consider  $[u]_v$  as a polynomial in  $k\langle X \rangle$ , then for any  $w \in \text{supp}([u]_v)$ ,  $\text{cont}(w) = \text{cont}(u)$ .

**Definition 3.11** Let  $S \subset \text{Lie}(X)$  with each  $s \in S$  monic,  $a, b \in X^*$  and  $s \in S$ . If  $a \bar{s} b$  is an ALSW, then we call  $[a s b]_{\bar{s}} = [a \bar{s} b]_{\bar{s}}|_{[\bar{s}] \mapsto s}$  a normal  $S$ -word (or normal  $s$ -word) while  $[a \bar{s} b]_{\bar{s}}$  is called a relative nonassociative Lyndon-Shirshov word, denoted by RNLSW, where  $[a \bar{s} b]_{\bar{s}}$  is defined by (3.1) (see Theorem 3.10).

**Corollary 3.12** Let  $u, v$  be ALSW's,  $f \in \text{Lie}(X)$ ,  $\bar{f} = v$  and  $u = a v b$ ,  $a, b \in X^*$ . Then, for the normal  $f$ -word  $[a f b]_v = [a v b]_v|_{[v] \mapsto f}$ , we have

$$[a f b]_v = a f b + \sum_i \alpha_i a_i f b_i,$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $a_i \bar{f} b_i < u$ .

## 4 Composition-Diamond lemma for associative algebras

In this Section, we cite some concepts and results from the literature which are related to the Gröbner-Shirshov basis for the associative algebras.

**Definition 4.1** ([17], see also [2], [3]) Let  $f$  and  $g$  be two monic polynomials in  $k\langle X \rangle$  and  $<$  a well order on  $X^*$ . Then, there are two kinds of compositions:

- (i) If  $w$  is a word such that  $w = \bar{f} b = a \bar{g}$  for some  $a, b \in X^*$  with  $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$ , then the polynomial  $(f, g)_w = f b - a g$  is called the intersection composition of  $f$  and  $g$  with respect to  $w$ .

(ii) If  $w = \bar{f} = a\bar{g}b$  for some  $a, b \in X^*$ , then the polynomial  $(f, g)_w = f - agb$  is called the inclusion composition of  $f$  and  $g$  with respect to  $w$ .

**Definition 4.2** ([2], [3], cf. [17]) Let  $S \subset k\langle X \rangle$  with each  $s \in S$  monic. Then the composition  $(f, g)_w$  is called trivial modulo  $(S, w)$  if  $(f, g)_w = \sum \alpha_i a_i s_i b_i$ , where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$  and  $\overline{a_i s_i b_i} < w$ . If this is the case, then we write

$$(f, g)_w \equiv_{ass} 0 \mod(S, w)$$

In general, for  $p, q \in k\langle X \rangle$ , we write

$$p \equiv_{ass} q \mod(S, w)$$

which means that  $p - q = \sum \alpha_i a_i s_i b_i$ , where  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$  and  $\overline{a_i s_i b_i} < w$ .

**Definition 4.3** ([2], [3], cf. [17]) We call the set  $S$  with respect to the well order  $<$  a Gröbner-Shirshov set (basis) in  $k\langle X \rangle$  if any composition of polynomials in  $S$  is trivial modulo  $S$ .

If a subset  $S$  of  $k\langle X \rangle$  is not a Gröbner-Shirshov basis, then we can add to  $S$  all nontrivial compositions of polynomials of  $S$ , and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis  $S^{comp}$ . Such a process is called the Shirshov algorithm.

A well order  $>$  on  $X^*$  is monomial if it is compatible with the multiplication of words, that is, for  $u, v \in X^*$ , we have

$$u > v \Rightarrow w_1 u w_2 > w_1 v w_2, \text{ for all } w_1, w_2 \in X^*.$$

A standard example of monomial order on  $X^*$  is the deg-lex order to compare two words first by degree and then lexicographically, where  $X$  is a linearly ordered set.

The following lemma was proved by Shirshov [17] for free Lie algebras (with deg-lex ordering) in 1962 (see also Bokut [2]). In 1976, Bokut [3] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For commutative polynomials, this lemma is known as the Buchberger's Theorem in [7] and [8].

**Lemma 4.4** (Composition-Diamond Lemma) Let  $k$  be a field,  $A = k\langle X | S \rangle = k\langle X \rangle / Id(S)$  and  $<$  a monomial order on  $X^*$ , where  $Id(S)$  is the ideal of  $k\langle X \rangle$  generated by  $S$ . Then the following statements are equivalent:

- (i)  $S$  is a Gröbner-Shirshov basis.
- (ii)  $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$  for some  $s \in S$  and  $a, b \in X^*$ .
- (ii')  $f \in Id(S) \Rightarrow f = \alpha_1 a_1 s_1 b_1 + \alpha_2 a_2 s_2 b_2 + \dots$ , where  $\alpha_i \in k$  and  $\bar{f} = a_1 \bar{s}_1 b_1 > a_2 \bar{s}_2 b_2 > \dots$ .
- (iii)  $Red(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$  is a basis of the algebra  $A = k\langle X | S \rangle$ .

## 5 Composition-Diamond lemma for Lie algebras

In this section, we give the Composition-Diamond lemma for Lie algebras.

Throughout this section, we extend the lexicographic order on  $X^*$  mentioned in Section 2 to the deg-lex order  $<$  on  $X^*$ .

**Lemma 5.1** *Let  $ac, cb$  be ALSW's, where  $a, b, c \in X^*$  and  $c \neq 1$ . Then  $w = acb$  is also an ALSW.*

*Proof.* We use induction on  $|w| = n$ . If  $n = 3$ , then  $w = x_i x_j x_k$  is an ALSW, because  $ac$  and  $cb$  are ALSW's implies that  $x_i > x_j > x_k$ . In the inductive case  $n > 3$ , suppose  $\min(w) = x_\beta$ ,  $b = x_\beta^e \tilde{b}$ ,  $e \geq 0$ ,  $\text{fir}(\tilde{b}) \neq x_\beta$  and  $\tilde{c} = cx_\beta^e$ . Then

$$w = a\tilde{c}\tilde{b} \quad \text{and} \quad w' = a'_w \tilde{c}'_w \tilde{b}'_w.$$

It is clear that  $a'_w \tilde{c}'_w, \tilde{c}'_w \tilde{b}'_w$  are ALSW's. By induction,  $w'$  is an ALSW and so is  $w$ .  $\square$

**Definition 5.2** *Let  $f$  and  $g$  be two monic Lie polynomials in  $\text{Lie}(X) \subset k\langle X \rangle$ . Then, there are two kinds of Lie compositions:*

- (i) *If  $w = \bar{f} = a\bar{g}b$  for some  $a, b \in X^*$ , then the polynomial  $\langle f, g \rangle_w = f - [agb]_{\bar{g}}$  is called the composition of inclusion of  $f$  and  $g$  with respect to  $w$ .*
- (ii) *If  $w$  is a word such that  $w = \bar{f}b = a\bar{g}$  for some  $a, b \in X^*$  with  $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$ , then the polynomial  $\langle f, g \rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}}$  is called the composition of intersection of  $f$  and  $g$  with respect to  $w$ .*

By Lemma 5.1, in the Definition 5.2 (i) and (ii),  $w$  is an ALSW.

**Definition 5.3** *Let  $S \subset \text{Lie}(X)$  be a nonempty subset,  $h$  a Lie polynomial and  $w \in X^*$ . We shall say that  $h$  is trivial modulo  $(S, w)$ , denoted by  $h \equiv_{\text{Lie}} 0 \pmod{(S, w)}$ , if  $h = \sum_i \alpha_i [a_i s_i b_i]_{\bar{s}_i}$ , where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$ ,  $[a_i \bar{s}_i b_i]_{\bar{s}_i}$  is a RNLSW and  $a_i \bar{s}_i b_i < w$ .*

**Definition 5.4** *Let  $S \subset \text{Lie}(X)$  be a nonempty set of monic Lie polynomials. Then  $S$  is called a Gröbner-Shirshov set (basis) in  $\text{Lie}(X)$  if any composition  $\langle f, g \rangle_w$  with  $f, g \in S$  is trivial modulo  $(S, w)$ , i.e.,  $\langle f, g \rangle_w \equiv_{\text{Lie}} 0 \pmod{(S, w)}$ .*

**Lemma 5.5** *Let  $f, g$  be monic Lie polynomials. Then*

$$\langle f, g \rangle_w - (f, g)_w \equiv_{\text{ass}} 0 \pmod{(\{f, g\}, w)}.$$

*Proof.* If  $\langle f, g \rangle_w$  and  $(f, g)_w$  are compositions of intersection, where  $w = \bar{f}b = a\bar{g}$ , then, by Corollary 3.12, we may assume that

$$\langle f, g \rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}} = fb + \sum_{I_1} \alpha_i a_i f b_i - ag - \sum_{I_2} \beta_j a_j g b_j,$$

where  $a_i \bar{f} b_i, a_j \bar{g} b_j < \bar{f}b = a\bar{g} = w$ . It follows that

$$\langle f, g \rangle_w - (f, g)_w \equiv_{\text{ass}} 0 \pmod{(\{f, g\}, w)}.$$

Similarly, for the case of the compositions of inclusion, we have the same conclusion.  $\square$

**Theorem 5.6** ([4],[5]) *Let  $S \subset \text{Lie}(X) \subset k\langle X \rangle$  be a nonempty set of monic Lie polynomials. Then  $S$  is a Gröbner-Shirshov basis in  $\text{Lie}(X)$  if and only if  $S$  is a Gröbner-Shirshov basis in  $k\langle X \rangle$ .*

*Proof.* Note that, by the definitions, for any  $f, g \in S$ , they have composition in  $\text{Lie}(X)$  if and only if so do in  $k\langle X \rangle$ .

Suppose that  $S$  is a Gröbner-Shirshov basis in  $\text{Lie}(X)$ . Then, for any composition  $\langle f, g \rangle_w$ , we have

$$\langle f, g \rangle_w = \sum_{I_1} \alpha_i [a_i s_i b_i]_{\bar{s}_i},$$

where  $[a_i \bar{s}_i b_i]_{\bar{s}_i}$  are RNLSW's and  $a_i \bar{s}_i b_i < w$ . By Corollary 3.12,

$$\langle f, g \rangle_w = \sum_{I_2} \beta_j c_j s_j d_j,$$

where each  $c_j \bar{s}_j d_j < w$ . Thus, by Lemma 5.5, we get

$$(f, g)_w \equiv_{ass} 0 \mod(S, w).$$

Hence,  $S$  is a Gröbner-Shirshov basis in  $k\langle X \rangle$ .

Conversely, assume that  $S$  is a Gröbner-Shirshov basis in  $k\langle X \rangle$ . Then, for any composition  $\langle f, g \rangle_w$  in  $S$ , by Lemma 5.5, we obtain

$$\langle f, g \rangle_w \equiv_{ass} (f, g)_w \equiv_{ass} 0 \mod(S, w).$$

Therefore, we can assume, by Lemma 4.4, that

$$\langle f, g \rangle_w = \sum_{I_1} \alpha_i a_i s_i b_i,$$

where  $a_i \bar{s}_i b_i < w$  and  $w > a_1 \bar{s}_1 b_1 > a_2 \bar{s}_2 b_2 > \dots$ . By noting that  $\langle f, g \rangle_w \in \text{Lie}(X)$ ,  $\overline{\langle f, g \rangle_w} = a_1 \bar{s}_1 b_1$  is an ALSW which shows that  $[a_1 \bar{s}_1 b_1]_{\bar{s}_1}$  is a RNLSW. Let  $h_1 = \langle f, g \rangle_w - \alpha_1 [a_1 \bar{s}_1 b_1]_{\bar{s}_1}$ . Clearly,  $\overline{h_1} < \overline{\langle f, g \rangle_w}$ . Then, by Corollary 3.12, we have

$$h_1 \equiv_{ass} 0 \mod(S, w).$$

Now, by induction on  $\overline{\langle f, g \rangle_w}$ , we have

$$\langle f, g \rangle_w = \sum_{I_2} \alpha_i [c_i s_i d_i]_{\bar{s}_i},$$

where each  $[c_i \bar{s}_i d_i]_{\bar{s}_i}$  is a RNLSW and  $c_i \bar{s}_i d_i < w$ . This proves that  $S$  is a Gröbner-Shirshov basis in  $\text{Lie}(X)$ .  $\square$

**Lemma 5.7** *Let  $S \subset \text{Lie}(X)$  with each  $s \in S$  monic. Let*

$$\text{Red}(S) = \{[u] \mid [u] \text{ is a NLSW, } u \neq a\bar{s}b, s \in S, a, b \in X^*\}.$$

*Then, for any  $h \in \text{Lie}(X)$ ,  $h$  has a representation:*

$$h = \sum_{[u_i] \in \text{Red}(S), u_i \leq \bar{h}} \alpha_i [u_i] + \sum_{s_j \in S, a_j \bar{s}_j b_j \leq \bar{h}} \beta_j [a_j s_j b_j]_{\bar{s}_j}.$$



*Proof.* We can assume that  $h = \sum_i \alpha_i [u_i]$ , where each  $[u_i]$  is a NLSW,  $0 \neq \alpha_i \in k$  and  $u_1 > u_2 > \dots$ . If  $[u_1] \in \text{Red}(S)$ , then let  $h_1 = h - \alpha_1 [u_1]$ . If  $[u_1] \notin \text{Red}(S)$ , then there exists  $s \in S$  and  $a_1, b_1 \in X^*$  such that  $u_1 = a_1 \bar{s}_1 b_1$ . Now, let

$$h_1 = h - \alpha_1 [a_1 s_1 b_1]_{\bar{s}_1} \in \text{Lie}(X).$$

Hence, in both cases, we have  $\bar{h}_1 < \bar{h}$ . Now, the result follows from induction on  $\bar{h}$ .  $\square$

**Theorem 5.8** *Let  $S \subset \text{Lie}(X) \subset k\langle X \rangle$  be nonempty set of monic Lie polynomials. Let  $\text{Id}_{\text{Lie}}(S)$  be the Lie-ideal of  $\text{Lie}(X)$  generated by  $S$ . Then the following statements are equivalent.*

- (i)  $S$  is a Gröbner-Shirshov basis in  $\text{Lie}(X)$ .
- (ii)  $f \in \text{Id}_{\text{Lie}}(S) \implies \bar{f} = a\bar{s}b$ , for some  $s \in S$  and  $a, b \in X^*$ .
- (ii')  $f \in \text{Id}_{\text{Lie}}(S) \implies f = \alpha_1 [a_1 s_1 b_1]_{\bar{s}_1} + \alpha_2 [a_2 s_2 b_2]_{\bar{s}_2} + \dots$ , where  $\alpha_i \in k$  and  $\bar{f} = a_1 \bar{s}_1 b_1 > a_2 \bar{s}_2 b_2 > \dots$ .
- (iii)  $\text{Red}(S) = \{[u] \mid [u] \text{ is a NLSW, } u \neq a\bar{s}b, s \in S, a, b \in X^*\}$  is a  $k$ -basis for  $\text{Lie}(X|S)$ .

*Proof.* (i)  $\implies$  (ii). By noting that  $\text{Id}_{\text{Lie}}(S) \subseteq \text{Id}_{\text{ass}}(S)$ , where  $\text{Id}_{\text{ass}}(S)$  is the ideal of  $k\langle X \rangle$  generated by  $S$ , and by using Theorem 5.6 and Lemma 4.4, the result follows.

(ii)  $\implies$  (iii). Suppose that  $\sum_{[u_i] \in \text{Red}(S)} \alpha_i [u_i] = 0$  in  $\text{Lie}(X|S)$  with  $u_1 > u_2 > \dots$ , that is,  $\sum_{[u_i] \in \text{Red}(S)} \alpha_i [u_i] \in \text{Id}_{\text{Lie}}(S)$ . Then each  $\alpha_i$  must be 0. Otherwise, say  $\alpha_1 \neq 0$ . Then, by (ii), we know that  $\overline{\sum_i \alpha_i [u_i]} = u_1$  which implies that  $[u_1] \notin \text{Red}(S)$ , a contradiction.

On the other hand, for any  $f \in \text{Lie}(X)$ , by Lemma 5.7, we have

$$f + \text{Id}_{\text{Lie}}(S) = \sum_i \alpha_i ([u_i] + \text{Id}_{\text{Lie}}(S)).$$

(iii)  $\implies$  (i). For any composition  $\langle f, g \rangle_w$  with  $f, g \in S$ , we have  $\langle f, g \rangle_w \in \text{Id}_{\text{Lie}}(S)$ . Then, by (iii) and by Lemma 5.7,

$$\langle f, g \rangle_w = \sum \beta_j [a_j s_j b_j]_{\bar{s}_j},$$

where each  $\beta_j \in k$ ,  $[a_j s_j b_j]_{\bar{s}_j}$  is normal  $S$ -word and  $a_j \bar{s}_j b_j < w$ . This proves that  $S$  is a Gröbner-Shirshov basis in  $\text{Lie}(X)$ .

(ii)  $\iff$  (ii'). This part is clear.  $\square$

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